Suggested solution of HW6

Q1 It suffices to consider nonnegative function f. Noted that for simple function $s = \sum_{i=1}^N \alpha_i \chi_{A_i}$, $\int_A s = \sum_{i=1}^N \alpha_i \mu(A \cap A_i)$.

Suppose s is a simple function on E such that $0 \le s \le f$ on A. Then $\tilde{s} = s \cdot \chi_A$ is a simple function on E in which

$$0 \le \tilde{s} \le f\chi_A, \qquad \text{on } \ E.$$

Thus,

$$\int_A s = \int_A \tilde{s} = \int_E \tilde{s} \le \sup\{\int_E s : 0 \le s \le f\chi_A\}$$

Taking sup against all such simple function yields

$$\sup\{\int_A s : 0 \le s \le f \ \text{ on } \ A\} \le \sup\{\int_E s : 0 \le s \le f\chi_A\}.$$

On the other hand, if s is a simple function defined on A such that $0 \le s \le f\chi_A$. By extending s to be zero elsewhere, we have the reverse inequality.

We only discuss the case when $f \ge 0$ and $f = g$ almost everywhere. The rest follows similarily. Let $s = \sum_{i=1}^N \alpha_i \chi_{A_i}$ be a simple function such that $0 \le s \le f$. Denote $B = \{f = g\}$, then $\tilde{s} = s \cdot \chi_B$ is still a simple function in which $0 \le \tilde{s} \le g$ and $\int s = \int \tilde{s}$. Hence,

$$\int g \ge \int \tilde{s} = \int s.$$

Taking sup over all such s yields $\int g \ge \int f$. By symmetric, equality follows.

Real Analysis HW6

Q2: Suppose E_i is a collection of disjoint measurable sets.

(a) Define $F_1 = E_1$, $F_2 = E_1 \cup E_2$,...,.$F_n = \cup_{k=1}^n E_k$. Then we have

$$\int_E f\chi_{F_n} = \int_{F_n} f = \sum_{k=1}^n \int_{E_k} f.$$

As $\chi_{F_n} \to \chi_{\cup E_k}$, we have the conclusion by MCT.

(b) Since $|f\chi_{F_n}| \le |f|$ on E, we have

$$\lim_{n\to\infty} \int_E f\chi_{F_n} = \int_E \lim_{n\to\infty} f\chi_{F_n} = \int_E f \cdot \chi_{\cup E_k} = \int_{\cup E_k} f.$$

And for each n,

$$\int_E f\chi_{F_n} = \sum_{k=1}^n \int_{E_k} f.$$

Q3: (a) Choose a function on $\mathbb{R}$ such that $f \ge -1$ and $\int_{\mathbb{R}} f = -\infty$. Then the function $f_n = \frac{1}{n} f$ converges to 0 everywhere. But the conclusion fails.

(b) Choose a function f on $\mathbb{R}$ such that $f > 0$ and $\int_{\mathbb{R}} f = \infty$ and $f_n = \frac{1}{n} f$.

(c) Choose a function $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 f = 1$. Define

$$f_n(x) = nf(nx).$$

Then $f_n(x) \to 0$ but the integral is constant.

(d) Let $f = \chi_{[0,1]}$ and $f_n(x) = f(x+n)$.

Q4: Denote $E_n = \{x : |f(x)| > n\}$.

$$n \cdot m(E_n) \le \int_{E_n} |f| < ||f||_{L^1}.$$

Hence, $m(\cap E_k) \le m(E_n) \le \frac{C}{n}$ for all n. Letting $n \to \infty$ to conclude that f is finite almost everywhere.

Noted that $|f_n - f| \le 2|f|$ for almost everywhere x and $f_n - f \to 0$ a.e. Hence, by DCT,

$$\int_E |f_n - f| \to 0.$$

It remains to show the uniform integrability. Suppose the conclusion fail. There exists $\epsilon > 0$ such that for any n, we can find a $A_n \subset E$ so that $m(A_n) < 2^{-n}$ but

$$\int_{A_n} |f| \ge \epsilon.$$

Define

$$A = \cap_{n=1}^\infty \cup_{k=n}^\infty A_k.$$

Then $m(A) = 0$ but by DCT

$$\int_A |f| \ge \epsilon.$$